Chapter 2

Change Of Variables

Let φ be a continuously differentiable function that maps $[\alpha, \beta]$ into $[a, b]$. For every continuous function f on $[a, b]$, we have following change of variables formula :

$$\int_{\alpha}^{\beta} f(\varphi(y))\varphi'(y)\,dy = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx\ . \tag{2.1}$$

The formula comes from a direct application of the Fundamental Theorem of Calculus. Let $F(x)$ be a primitive function of f, that is, $F' = f$. Consider the composite function $g(y) = F(\varphi(y))$. By the chain rule,

$$g'(y) = F'(\varphi(y))\varphi'(y) = f(\varphi(y))\varphi'(y)\ .$$

By the fundamental theorem of calculus,

$$g(\beta) - g(\alpha) = \int_{\alpha}^{\beta} g'(y)\,dy = \int_{\alpha}^{\beta} f(\varphi(y))\varphi'(y)\,dy\ .$$

On the other hand,

$$g(\beta) - g(\alpha) = F(\varphi(\beta)) - F(\varphi(\alpha)) = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx\ .$$

Hence the formula holds.

When φ maps $[\alpha, \beta]$ bijectively onto $[a, b]$, either φ is strictly increasing with $\varphi(\alpha) = a,\ \varphi(\beta) = b$ or it is strictly decreasing with $\varphi(\alpha) = b,\ \varphi(\beta) = a$. In the first case φ' is non-negative or in the second case non-positive. So (2.1) becomes the formula

$$\int_{\alpha}^{\beta} f(\varphi(y))|\varphi'(y)|\,dy = \int_{a}^{b} f(x)\,dx\ . \tag{2.2}$$

In the first two sections we will extend (1.2) to higher dimension. In the last two sections we consider an extension of (1.1).

2.1 The Change Of Variables Formula

Let D_1 and D_2 be two regions in $\mathbb{R}^n$. (Here we are mainly concerned with $n = 2, 3$.) A bijective map from D_1 to D_2 is called a C^1-*diffeomorphism* if it and its inverse are both continuously differentiable.

For a differentiable map Φ from D_1 to $\mathbb{R}^n$, its *Jacobian matrix* $\nabla\Phi$ is given by $(\partial\Phi_i/\partial x_j), i, j = 1, 2, \cdots, n$, that is,

$$\begin{bmatrix} \dfrac{\partial \Phi_1}{\partial x_1} & \cdots & \cdots & \dfrac{\partial \Phi_1}{\partial x_n} \\ \dfrac{\partial \Phi_2}{\partial x_1} & \cdots & \cdots & \dfrac{\partial \Phi_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial \Phi_n}{\partial x_1} & \cdots & \cdots & \dfrac{\partial \Phi_n}{\partial x_n} \end{bmatrix}$$

The determinant of the Jacobian matrix is called the *Jacobian* of Φ. It will be denoted by J_Φ.

By the Inverse Function Theorem, a C^1-map from a region D in $\mathbb{R}^n$ to $\mathbb{R}^n$ which is one-to-one and whose Jacobian never vanishes sets up a C^1-diffeomorphism between D and its image $\Phi(D)$. This fact will be used implicitly and frequently below.

Theorem 2.1. ***(Change of Variables Formula)*** *Let Φ be a C^1-diffeomorphism from D_1 to D. For any continuous function f in D,*

$$\int_D f(\boldsymbol{x})\, d\boldsymbol{x} = \int_{D_1} f(\Phi(\boldsymbol{y}))|J_\Phi(\boldsymbol{y})|\, d\boldsymbol{y}\ . \tag{2.3}$$

Here $d\mathbf{x}$ and $d\mathbf{y}$ refer to the integration over an n-dimensional region. For $n = 2$, in our usual notation, this formula reads as,

$$\iint_D f(x, y)\, dA(x, y) = \iint_{D_1} f(\Phi(u, v))|J_\Phi(u, v)|\, dA(u, v)\ ,$$

and for $n = 3$,

$$\iiint_\Omega f(x, y, z)\, dV(x, y, z) = \iiint_{\Omega_1} f(\Phi(u, v, w))|J_\Phi(u, v, w)|\, dV(u, v, w)\ .$$

The integration formulas for the polar coordinates, cylindrical coordinates and spherical coordinates are special cases of this theorem.

In the case of the polar coordinates, we take $n = 2$ and $\Phi(r, \theta) = (r\cos\theta, r\sin\theta)$. Then $J_\Phi = r \geq 0$, so the formula (2.3) becomes

$$\iint_D f(x, y)\, dA(x, y) = \iint_{D_1} f(r\cos\theta, r\sin\theta) r\, dA(r, \theta)\ .$$

In the case of the cylindrical coordinates, we take $n = 3$ and $\Phi(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$. Then $J_\Phi = r$ and (2.3) becomes

$$\iiint_\Omega f(x, y, z)\, dV = \iint_{\Omega_1} f(r\cos\theta, r\sin\theta, z) r\, dV(r, \theta, z)\ .$$

when

In the case of the spherical coordinates, we take $n = 3$ and

$$\Phi(\rho, \varphi, \theta) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\ , \quad \varphi \in [0, \pi],\ \theta \in [0, 2\pi)\ .$$

Then $J_\Phi = \rho^2 \sin\varphi \geq 0$ and (2.3) becomes

$$\iiint_\Omega f(x, y, z)\, dV = \iiint_{\Omega_1} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2 \sin\varphi\, dV(\rho, \varphi, \theta)\ .$$

We now explain the ideas behind (2.3).

We take $n = 2$ and D_1 a rectangle. A partition $P = \{R_{ij}\}$ on D_1 introduces a generalized partition $\{D_{ij}\}$ on D. Now, for a continuous function f in D, when the partition P becomes very fine, by Theorem 1.10,

$$\begin{aligned} \iint_D f\, dA &\approx \sum_{i,j} f(p_{ij})|D_{ij}| \\ &= \sum_{i,j} f(\Phi(q_{ij}))\frac{|D_{ij}|}{|R_{ij}|}|R_{ij}|\ , \end{aligned}$$

where p_{ij} is a tag point in D_{ij} and $\Phi(q_{ij}) = p_{ij}$. This is possible because Φ is bijective.

Now, let us focus on a subrectangle R_{ij}. Let $(u, v), (u+h, v), (u, v+k), (u+h, v+k)$ be the vertices of the subrectangle. (We have dropped the subscripts i, j for simplicity. (u, v) should be (u_i, v_j) and $h = \Delta x_i, k = \Delta y_j$.) Its image D_{ij} has vertices at $\Phi(u, v), \Phi(u + h, v), \Phi(u, v + k)$, and $\Phi(u + h, v + k)$. By Taylor's expansion,

$$\Phi(u + h, v) = \Phi(u, v) + \Phi_u(u, v)h + \text{ higher order terms},$$

$$\Phi(u, v+k) = \Phi(u,v) + \Phi_v(u,v)k + \text{ higher order terms},$$

and

$$\Phi(u+h, v+k) = \Phi(u,v) + \Phi_u(u,v)h + \Phi_v(u,v)k + \text{ higher order terms} .$$

Ignoring the higher order terms, D_{ij} is well approximated by the parallelogram with vertices at $\Phi(u,v), \Phi(u,v) + \Phi_u(u,v)h, \Phi(u,v) + \Phi_v(u,v)k,$ and $\Phi(u,v) + \Phi_u(u,v)h + \Phi_v(u,v)k$. Recall that for a parallelogram spanned by two vectors (a_1, a_2) and (b_1, b_2), its area is given by $|a_1b_2 - a_2b_1|$. Therefore, the area of our parallelogram is equal to $|J_\Phi(u,v)|hk$. As hk is just the area of R_{ij}, so

$$\frac{|D_{ij}|}{|R_{ij}|} \approx \frac{|J_\Phi(u_i, v_j)|hk}{hk} = |J_\Phi(u_i, v_j)|.$$

It follows that

$$\sum_{i,j} f(\Phi(q_{ij}))\frac{|D_{ij}|}{|R_{ij}|}|R_{ij}| \approx \sum_{i,j} f(\Phi(q_{ij}))|J_\Phi(u_i, v_j)||R_{ij}| .$$

Note that (u_i, v_j) is also a tag point in R_{ij}. Applying Theorem 1.11, as $\|P\| \to 0$,

$$\iint_D f(x,y)\, dA(x,y) = \iint_{D_1} f(\Phi(u,v))|J_\Phi|(u,v)\, dA(u,v) .$$

Similarly, in $n = 3$, the subrectangular box B_{ijk} maps to a parallelepiped Ω_{ijk} under Φ and the volume ratio

$$\frac{|\Omega_{ijk}|}{|B_{ijk}|} \approx |J_\Phi(u_i, v_j, w_k)| .$$

In the following we look at some examples. We point out that in $n = 2, 3$, people like to use another notation for the Jacobian matrix, for instance, J_Φ is written as

$$\frac{\partial(x,y)}{\partial(u,v)} .$$

The variables in the numerator and denominator are respective the dependent and independent variables. In the next section we will establish the useful relation:

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} .$$

Example 2.1. Find the area of the region bounded by the curves $y = x, y = 6x, xy = 1$ and $xy = 5$.

We make the region simpler by introducing the change of variables $u = y/x$ and $v = xy$. The rectangle $(u, v) \in [1, 6] \times [1, 5]$ is mapped to the region under $\Phi : (u, v) \mapsto (x, y)$. The map Φ can be determined by expressing x, y in terms of u, v. After a little manipulation, we get $x = \sqrt{vu^{-1}}, y = \sqrt{uv}$. The Jacobian is equal to $1/(-2u)$. It follows that the area is given by

$$\begin{aligned}
\iint_D 1\, dxdy &= \int_1^6 \int_1^5 \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv \\
&= \int_1^6 \int_1^5 \left| \frac{1}{-2u} \right| dvdu \\
&= 2 \log 6\ .
\end{aligned}$$

We point out one can determine the Jacobian without Φ. Indeed, the Jacobian of the inverse map is

$$\frac{\partial(u, v)}{\partial(x, y)} = -2y/x = -2u.$$

By the relation above, the Jacobian of Φ is $1/(-2u)$.

Example 2.2. Evaluate the iterated integral

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}(y - 2x)^2\, dydx\ .$$

This is a double integral over the triangle with vertices at $(0, 0), (1, 0)$ and $(0, 1)$. While the region of integration is simple enough, the integrand is a bit messy. Unlike the first example, we simplify the integrand this time. Letting $u = x + y$ and $v = y - 2x$, the integrand becomes $\sqrt{u}v^2$ but the region becomes the region bounded by the curves $x = 0, y = 0, x + y = 1$ which go over to $u = v, 2u + v = 0$ and $u = 1$. The Jacobian of the inverse map is

$$\frac{\partial(u, v)}{\partial(x, y)} = 3\ .$$

Therefore,

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \sqrt{x+y}(y - 2x)^2\, dydx &= \int_0^1 \int_{-2u}^{u} \sqrt{u}v^2 \frac{1}{3}\, dydx \\
&= \frac{2}{9}\ .
\end{aligned}$$

Example 2.3 Evaluate
$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}} e^{\sqrt{xy}}\, dxdy \ .$$

The region is composed three sides given by $y = x, xy = 1$ and $y = 2$. Or,
$$D = \{(x, y) : 1/y \le x \le y,\ y \in [1, 2]\}.$$
Let $u = \sqrt{xy}$ and $v = \sqrt{y/x}$ or $x = u/v, y = uv$. The region goes over to the region bounded by $v = 1, u = 1$ and $uv = 2$. Or,
$$D_1 = \{(u, v) :\ 1 \le v \le 2/u,\ v \in [1, 2]\}.$$
We have
$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{2u}{v} \ .$$
Therefore, our integral is equal to
$$\int_1^2 \int_1^{2/u} ve^u \frac{2u}{v}\, dvdu = 2e(e - 2) \ .$$

Next we look at some three dimensional examples.

Example 2.4 Evaluate
$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=y/2+1} \left(\frac{2x - y}{2} + \frac{z}{3}\right) dxdydz \ .$$

The region projected to the rectangle $[0, 3] \times [0, 4]$ in yz-plane and is simple enough. Let $t = x - y/2 \in [0, 1], y = y, z = z$ be the change of variables. The Jacobian is equal to 1. Therefore, this integral is equal to
$$\int_0^3 \int_0^4 \int_0^1 \left(t + \frac{z}{3}\right) dtdydz = 12 \ .$$

Example 2.5. Find the volume of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1$.

Introducing the change of variables $x = au, y = bv, z = cw$, the ellipsoid is the image of the unit ball B, $u^2 + v^2 + w^2 \le 1$. We have
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = abc \ .$$
Therefore, the volume of the ellipsoid is given by
$$\iiint_B 1 \times abc\, dV(u, v, w) = \frac{4}{3}\pi abc \ .$$

2.2 Proof Of The Formula

In this section we present a more detailed proof of (2.3). We will follow Prof Tom Wan's treatment. Further discussions can be found in M. Spivak's book: *Calculus on Manifolds.* To prepare for it, we need the following propositions.

Proposition 2.2. *Let* $\Phi_1 : D_2 \to D_1$ *and* $\Phi_2 : D_3 \to D_2$ *be two* C^1*-maps. Then* $\Phi = \Phi_1 \circ \Phi_2$ *satisfies*

$$\nabla\Phi = \nabla\Phi_1 \cdot \nabla\Phi_2 \ ,$$

(matrix product) and

$$J_\Phi = J_{\Phi_1} J_{\Phi_2} \ .$$

Proof. Let $\mathbf{x} = \Phi_1(\mathbf{y})$ and $\mathbf{y} = \Phi_2(\mathbf{z})$ so that $\mathbf{x} = \Phi(\mathbf{z})$. We have

$$\Phi_i(\mathbf{z}) = (\Phi_1 \circ \Phi_2)_i(\mathbf{z}) = \Phi_{1i}(\Phi_{21}(\mathbf{z}), \cdots, \Phi_{2n}(\mathbf{z})) \ .$$

By the Chain Rule,

$$\frac{\partial \Phi_i}{\partial z_j}(\mathbf{z}) = \sum_k \frac{\partial \Phi_{1i}}{\partial y_k}(\mathbf{y}) \frac{\partial \Phi_{2k}}{\partial z_j}(\mathbf{z}) \ ,$$

which is precisely the matrix product

$$\nabla\Phi(\mathbf{z}) = \nabla\Phi_1(\mathbf{y}) \cdot \nabla\Phi_2(\mathbf{z}) \ .$$

The second formula follows from the property of the determinant: $\det AB = \det A \det B$. □

Proposition 2.3. *Let* $\Phi : D_1 \to D$ *be a* C^1*-diffeomorphism. Then*

$$J_{\Phi^{-1}} J_\Phi = 1 \ .$$

In particular, $J_\Phi \neq 0$ *in* D_1*.*

Proof. We have $\Phi^{-1}(\Phi(\mathbf{x})) = \mathbf{x}$. By Proposition 2.2 and using the fact that the Jacobian matrix of the identity map is the identity matrix, $\nabla\Phi^{-1} \cdot \nabla\Phi$ is equal to the identity matrix, and the formula follows by taking determinant of the both sides. □

Proposition 2.4. *Let* $\Phi_1 : D_2 \to D_1$ *and* $\Phi_2 : D_3 \to D_2$ *be two* C^1*-diffeomorphisms and* $\Phi = \Phi_1 \circ \Phi_2 : D_3 \to D_1$*. Suppose (1.3) holds for* Φ_1 *and* Φ_2*. Then it also holds for* Φ*.*

Proof. Let f and g be continuous in D_1 and D_2 respectively. By assumption, we have

$$\int_{D_1} f(\mathbf{x})\, d\mathbf{x} = \int_{D_2} f(\Phi_1(\mathbf{y}))|J_{\Phi_1}|(\mathbf{y})\, d\mathbf{y} \ ,$$

and

$$\int_{D_2} g(\mathbf{y})\, d\mathbf{y} = \int_{D_3} g(\Phi_2(\mathbf{z}))|J_{\Phi_2}|(\mathbf{z})\, d\mathbf{z}\ .$$

As $f(\Phi_1(\mathbf{y}))|J_{\Phi_1}|(\mathbf{y})$ is continuous in D_2, taking it to be g, we have

$$\begin{aligned}\int_{D_1} f(\mathbf{x})\, d\mathbf{x} &= \int_{D_2} f(\Phi_1(\mathbf{y}))|J_{\Phi_1}|(\mathbf{y})\, d\mathbf{y} \\ &= \int_{D_3} f(\Phi_1(\Phi_2(\mathbf{z})))|J_{\Phi_1}(\Phi_2(\mathbf{z}))||J_{\Phi_2}|(\mathbf{z})\, d\mathbf{z} \\ &= \int_{D_3} f(\Phi(\mathbf{z}))|J_\Phi|(\mathbf{z})\, d\mathbf{z}\ . \quad (\text{ Proposition 2.2})\end{aligned}$$

□

Now let us restrict to $n = 2$.

Proposition 2.5. *The change of variables formula (1.3) holds in the following two cases:*

(a) Φ *is a* C^1*-diffeomorphism of the form* $\Phi(u, v) = (\varphi(u, v), v)$ *in* D*; and*

(b) Φ *is a* C^1*-diffeomorphism of the form* $\Phi(u, v) = (u, \psi(u, v))$*.*

Proof. We prove (a) while (b) can be proved in a similar way. We will take D to be a rectangle $[a, b] \times [c, d]$. First of all, the Jacobian of Φ is equal to $\partial\varphi/\partial u$. By Proposition 2.3, either $\partial\varphi/\partial u > 0$ or $\partial\varphi/\partial u < 0$ throughout D. Assume it is the former. Under Φ, the vertical line (u, v), where $u \in [a, b]$ is fixed, is mapped to $(\varphi(u, v), v)$. This is the graph of $\varphi(u, \cdot)$ over $[c, d]$. Since for each fixed v, $\varphi_u > 0$, $\varphi(u_1, v) < \varphi(u_2, v)$ for $u_1 < u_2$, the image of D under φ is of the form:

$$\{(x, y) :\ \varphi(a, y) \le x \le \varphi(b, y),\ y = v \in [c, d]\}.$$

By Fubini's theorem,

$$\begin{aligned}\iint_{\Phi(D)} f(x, y)\, dA &= \int_c^d \int_{\varphi(a,y)}^{\varphi(b,y)} f(x, y)\, dxdy \\ &= \int_c^d \int_a^b f(\varphi(u, y), y)\frac{\partial\varphi}{\partial u}\, dudy\ , \\ &= \int_c^d \int_a^b f(\varphi(u, v), v)\frac{\partial\varphi}{\partial u}\, dudv \\ &= \iint_D f(\Phi(u, v))|J_\Phi|(u, v)\, dA(u, v)\ .\end{aligned}$$

Note that at the second line, we have used the change of variables $x = \varphi(u, y)$ so that $dx = \varphi_u du$.

When $\partial\varphi/\partial u < 0$, $\Phi(D)$ becomes

$$\{(x,y):\ \varphi(b,y) \le x \le \varphi(a,y),\ y = v \in [c,d]\}.$$

Similarly as above, we have

$$\begin{aligned}\iint_{\Phi(D)} f(x,y)\,dA &= \int_c^d \int_{\varphi(b,y)}^{\varphi(a,y)} f(x,y)\,dxdy \\ &= \int_c^d \int_b^a f(\varphi(u,y),y)\frac{\partial\varphi}{\partial u}\,dudy\ , \\ &= \int_c^d \int_a^b f(\varphi(u,v),v)\left|\frac{\partial\varphi}{\partial u}\right|\,dudv \\ &= \iint_D f(\Phi(u,v))|J_\Phi|(u,v)\,dA(u,v)\ .\end{aligned}$$

□

Now we prove the Change of Variables Formula (1.3) for a general Φ. For simplicity we will only consider $n = 2$ and take D_1 to be a rectangle. The general case is essentially the same. First of all, since $\Phi = (\varphi_1, \varphi_2)$ is a C^1-diffeomorphism, its Jacobian

$$J_\Phi = \frac{\partial\varphi_1}{\partial u}\frac{\partial\varphi_2}{\partial v} - \frac{\partial\varphi_1}{\partial v}\frac{\partial\varphi_2}{\partial u}$$

never vanishes. Therefore, either $\partial\varphi_1/\partial u$ or $\partial\varphi_1/\partial v$ is not zero at any point. We can fix a partition on D so fine that each subrectangle R_{ij} belongs to either $\mathcal{A}$ or $\mathcal{B}$ where

$$\mathcal{A} = \{R_{ij}:\ \frac{\partial\varphi_1}{\partial u} > 0 \text{ in } R_{ij}\},$$

and

$$\mathcal{B} = \{R_{ij}:\ \frac{\partial\varphi_1}{\partial v} < 0 \text{ in } R_{ij}\}\ .$$

Using

$$\begin{aligned}&\iint_D f(\Phi(u,v))|J_\Phi|(u,v)\,dA \\ =\ &\sum_{R_{ij}\in\mathcal{A}} \iint_{R_{ij}} f(\Phi(u,v))|J_\Phi|(u,v)\,dA + \sum_{R_j\in\mathcal{B}} \iint_{R_{ij}} f(\Phi(u,v))|J_\Phi|(u,v)\,dA\ ,\end{aligned}$$

we see that it suffices to establish the formula under the additional assumption $R \in \mathcal{A}$ or $R \in \mathcal{B}$. (We have written R for R_{ij} for simplicity.)

Let us only consider $R \in \mathcal{A}$. (The other case can be handled in a similar way.) We consider the maps $\Phi_1(u,v) = (\varphi_1(u,v), v)$ and $\Phi_2(s,t) = (s, h(s,t))$ where $h(s,t) = \varphi_2(\Phi_1^{-1}(s,t)$. Since $J_{\Phi_1} = \partial\varphi_1/\partial u \neq 0$, Φ_1 is a C^1-diffeomorphism from R onto its image. In particular, the inverse map Φ_1^{-1} exists. Now,

$$\begin{aligned}
\Phi_2(\Phi_1(u,v)) &= \Phi_2(\varphi_1(u,v), v) \\
&= (\varphi_1(u,v), h(\varphi_1(u,v), v)) \\
&= (\varphi_1(u,v), h(\Phi_1(u,v)) \\
&= (\varphi_1(u,v), \varphi_2(u,v)) \\
&= \Phi(u,v) \ .
\end{aligned}$$

By Propositions 2.4 and 2.5 , we see that (2.3) holds for Φ.

2.3 A Different Extension

In this section we present a different extension of the change of variables formula (2.1) to higher dimensions. It applies to a restricted class of functions.

Theorem 2.6. *Let Φ be a C^1-map from $\mathbb{R}^n$ to $\mathbb{R}^n, n \geq 2$, such that $\Phi(\boldsymbol{x}) = \boldsymbol{x}$ for all $\boldsymbol{x}$, $|\boldsymbol{x}| \geq R$ for some number R. For every continuous function f which vanishes outside some bounded set,*

$$\int_{\mathbb{R}^n} f(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\mathbb{R}^n} f(\Phi(\boldsymbol{y}))J_\Phi(\boldsymbol{y})\, d\boldsymbol{y} \ .$$

The main difference between this theorem and Theorem 2.1 is that now there is no need to take the absolute value of the Jacobian. Note that since f vanishes outside some bounded set, the integration is in fact over a large rectangle; it is not an improper integral.

We will prove this theorem for the special case $n = 2$, that is,

$$\iint_{\mathbb{R}^2} f(x,y)\, dA(x,y) = \iint_{\mathbb{R}^2} f(\Phi(u,v))J_\Phi(u,v)\, dA(u,v) \ . \tag{2.4}$$

The proof of the general case is essentially the same, see

P. Lax, *Change of variables in multiple integrals*, The American Mathematical Monthly, vol 106, 497-501, 2013.

Proof. We write $\Phi(u,v) = (x(u,v), y(u,v))$. To start, let us fix some large $a > 0$ such that Φ becomes the identity map and f vanishes outside the square $S = [-a,a] \times [-a,a]$. Define

$$g(x,y) = \int_{-a}^{y} f(x,t)\, dt \ ,$$

so that

$$\frac{\partial g}{\partial y} = f(x, y) \ .$$

Letting $\Phi(u, v) = (x(u, v), y(u, v))$, we have

$$\begin{aligned}
& \iint_{\mathbb{R}^2} f(\Phi(u,v)) J_\Phi \, dA(u,v) \\
= & \iint_S f(\Phi(u,v)) J_\Phi \, dA(u,v) \\
= & \int_{-a}^{a}\int_{-a}^{a} \frac{\partial g}{\partial y}(x(u,v), y(u,v)) \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} dudv \\
= & \int_{-a}^{a}\int_{-a}^{a} \det \begin{bmatrix} x_u & x_v \\ g_y y_u & g_y y_v \end{bmatrix} dudv \\
= & \int_{-a}^{a}\int_{-a}^{a} \det \begin{bmatrix} x_u & x_v \\ g_y y_u + g_x x_u & g_y y_v + g_x x_v \end{bmatrix} dudv \\
= & \int_{-a}^{a}\int_{-a}^{a} \det \begin{bmatrix} x_u & x_v \\ \frac{\partial}{\partial u} g(x(u,v), y(u,v)) & \frac{\partial}{\partial v} g(x(u,v), y(u,v)) \end{bmatrix} dudv \\
= & \int_{-a}^{a}\int_{-a}^{a} \left(x_u \frac{\partial}{\partial v} g(x(u,v), y(u,v)) - x_v \frac{\partial}{\partial u} g(x(u,v), y(u,v)) \right) dudv \ .
\end{aligned}$$

Now,

$$\begin{aligned}
& \int_{-a}^{a}\int_{-a}^{a} x_u \frac{\partial}{\partial v} g(x(u,v), y(u,v)) \, dudv \\
= & \int_{-a}^{a}\int_{-a}^{a} x_u \frac{\partial}{\partial v} g(x(u,v), y(u,v)) \, dvdu \\
= & -\int_{-a}^{a}\int_{-a}^{a} x_{uv} g(x(u,v), y(u,v)) \, dudv + \int_{-a}^{a} x_u g(x(u,v), y(u,v)) \Big|_{v=-a}^{v=a} du \ .
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \int_{-a}^{a}\int_{-a}^{a} x_v \frac{\partial}{\partial u} g(x(u,v), y(u,v)) \, dudv \\
= & -\int_{-a}^{a}\int_{-a}^{a} x_{uv} g(x(u,v), y(u,v)) \, dudv + \int_{-a}^{a} x_v g(x(u,v), y(u,v)) \Big|_{u=-a}^{u=a} du \ .
\end{aligned}$$

As Φ is equal to the identity on the boundary of S, in particular we have $(x(u, \pm a), y(u, \pm a)) = (u, \pm a)$ on the two horizontal sides of S. It follows that $x_u(u, \pm a) = 1$. On the other hand, $(x(\pm a, v), y(\pm a, v)) = (\pm a, v)$ on the two verical sides of S, hence $x(\pm a, v) = \pm a$

and $x_v(\pm a, v)) = 0$. Therefore,

$$
\begin{aligned}
& \iint_{\mathbb{R}^2} f(\Phi(u,v)) J_\Phi \, dA(u,v) \\
= & -\int_{-a}^{a}\int_{-a}^{a} x_{uv} g(x(u,v),y(u,v))\, dudv + \int_{-a}^{a} x_u g(x(u,v),y(u,v))\Big|_{v=-a}^{v=a} du \\
& + \int_{-a}^{a}\int_{-a}^{a} x_{uv} g(x(u,v),y(u,v))\, dudv + \int_{-a}^{a} x_v g(x(u,v),y(u,v))\Big|_{u=-a}^{u=a} du \\
= & \int_{-a}^{a} x_u g(x(u,v),y(u,v))\Big|_{v=-a}^{v=a} du \\
= & \int_{-a}^{a} (g(u,a) - g(u,-a))\, du \\
= & \int_{-a}^{a} g(u,a)\, du \quad (\text{as } g(u,-a) = 0) \\
= & \int_{-a}^{a}\int_{-a}^{a} f(u,t)\, dt \\
= & \iint_S f(x,y)\, dA(x,y) \\
= & \iint_{\mathbb{R}^2} f(x,y)\, dA(x,y) \ .
\end{aligned}
$$

□

In one step the second partial derivative x_{uv} is involved, but it can be removed easily by an approximation argument.

A consequence of this theorem is

Proposition 2.7. *The map Φ in Theorem 2.6 maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.*

Proof. Suppose not, the image of $\mathbb{R}^n$ under Φ is not the entire $\mathbb{R}^n$. Since Φ is the identity map outside the ball $B_R(\mathbf{0})$, the image must miss at least a point $\mathbf{x}_0, |\mathbf{x}_0| < R$. By the continuity of Φ, actually there is a small ball $B(\mathbf{x}_0)$ which is not contained in the image, that is, $\Phi(\mathbb{R}^n) \cap B(\mathbf{x}_0) = \phi$. We pick a continuous function g which is positive inside $B(\mathbf{x}_0)$ but vanishes outside $B(\mathbf{x}_0)$. By Theorem 2.6,

$$\int_{\mathbb{R}^n} g(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}^n} g(\Phi(\mathbf{y})) J_\Phi(\mathbf{y})\, d\mathbf{y} \ .$$

The left hand side of this formula

$$\int_{\mathbb{R}^n} g(\mathbf{x})\, d\mathbf{x} = \int_{B(x_0)} g(\mathbf{x})\, d\mathbf{x} > 0 \ .$$

However, its right hand side vanishes because $\Phi(\mathbb{R}^n) \cap B(\mathbf{x}_0) = \phi$ and g vanishes outside $B(\mathbf{x}_0)$, contradiction holds. □

2.4 Brouwer's Fixed Point Theorem

A nice application of the previous extension is a proof of Brouwer's fixed point theorem. This fundamental theorem was proved first by Brouwer using algebraic topology in 1911 and was hailed as a triumph of this new branch of mathematics. Nowadays, we know it could also be proved by analytic methods.

Theorem 2.8. ***(Brouwer's Fixed Point Theorem)*** *Let B be the ball $\{\boldsymbol{x} : |\boldsymbol{x}| \leq 1\}$ in $\mathbb{R}^n$. A continuous map $G : B \to B$ admits a fixed point, that is, there is some $\boldsymbol{z} \in B$ such that $G(\boldsymbol{z}) = \boldsymbol{z}$.*

Remarks 2.1.

(a). Consider a rotation on B in the plane. Clearly, the origin is its only fixed point. On the other hand, the reflection $(x, y) \to (x, -y)$ has the set $\{(x, 0), x \in [-1, 1]\}$ to be its fixed point set.

(b). Let D be a region which can be mapped onto the ball by a continuous bijective map H. (That is, D is *homeomorphic* to the ball.) For a continuous map Φ on D to D, the map $H \circ \Phi \circ H^{-1}$ is a continuous map on B to B. One readily checks that $H^{-1}(\mathbf{z})$ is a fixed point of Φ whenever $\mathbf{z}$ is a fixed point for $H \circ \Phi \circ H^{-1}$. Hence the property of having a fixed point is preserved under any homeomorphism. In other word, it is a "topological property".

(c). Any rotation on the annulus $\{x : 1 \leq |x| \leq 2\}$ does not admit any fixed point. It is obvious that the annulus cannot be homeomorphic to the ball.

Proof. Suppose on the contrary, there is a continuous map G on B to itself which does not admit any fixed point, that is, $G(\mathbf{x}) \neq \mathbf{x}$ for all $\mathbf{x} \in B$. For a point $\mathbf{x}$ lying in the interior of B, the line segment connecting $G(\mathbf{x})$ to $\mathbf{x}$ can be extended and hits the boundary of B at a point $\mathbf{y}$. When $\mathbf{x}$ lies on the boundary of B, set $\mathbf{y} = \mathbf{x}$. Then the map $\mathbf{x} \mapsto \mathbf{y}$ forms a continuous map from B to ∂B, the boundary of B, and is equal to the identity on ∂B. We extend this map to the outside of B to be the identity map. In this way, we obtain a continuous map Φ from $\mathbb{R}^n$ to $\mathbb{R}^n$ which misses the interior of B, but this is contradictory to Proposition 2.7. We conclude that G must admit at least one fixed point. □

A careful reader may find a gap in the proof above: The map Φ is only continuous, while in order to apply Proposition 2.7 one needs Φ to be C^1. Again this defect can be remedied by some approximation arguments. We will not dwell on this point.